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Space-charge waves carried by a plasma trapped in a potential well

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A mono-energetic electron plasma is trapped in a one-dimensional parabolic potential well. In the undisturbed state a suitable background charge provides space-charge neutrality. It is shown that stationary disturbances of infinitesimal amplitude can exist in the plasma for certain critical values of the parameter $\Lambda = 4\bar{n}e^2/m\omega^2$, where $\bar{n}$ is the mean electron density and $\omega/2\pi$ is the frequency of oscillation of the electrons in the well. The first few critical values are $\Lambda = 0$, 4·12, 8·2.

The boundary conditions at the end of the plasma are non-linear. As a result stationary disturbances of finite amplitude in a given mode, say the rth, require that Λ shall exceed Λ_r. Further it can be shown that disturbances of small amplitude in the rth mode are unstable when Λ exceeds Λ_r. This applies even for $\Lambda_r = 0$; in this case there exist nearby an unstable even and an unstable odd mode.

It seems likely that these results can be extended to all cases in which the potential well is symmetrical.

1. Introduction

Problems involving plasma waves are usually relatively easy to solve if the medium in which they are carried is infinite, and uniform in space and time. But the plasma is often finite and non-uniform in practical cases, for example, in the Phoenix experiment (Kuo, Murphy, Petravic & Sweetman 1963). It is therefore interesting to consider how finiteness or non-uniformity changes the plasma properties. Watson & Rowlands (1963) have recently described the nature of the two-stream instability in a plasma which is trapped in a parabolic potential well. Their solution applies to disturbances with many nodes between the end-points of the plasma. In this limit the wavelength of the disturbance is small compared with the typical scale of the non-uniformity. Watson & Rowlands find that the properties of the plasma waves are then altered only very little.

The same problem is tackled again in this paper. This time no restriction is placed on the number of nodes. We shall find stability criteria for disturbances in the various modes; it also turns out that there are non-linear effects present which cannot be neglected even for disturbances of small amplitude. They arise because, in a mono-energetic plasma, the electron density tends to infinity at the edges of the trap.

2. The linearized equations

When undisturbed let the charged particles move in a parabolic potential well between the planes $x = \pm a$ under a restoring force $-\omega^2 x$ per unit mass. The particles then oscillate in the well with a period $2\pi/\omega$. If n_0 is the electron density at $x = 0$, and $\bar{n}$ the mean electron density in the trap, then

$$2\bar{n}a\omega/\pi = n_0 u_0 = n_0 \omega a, \tag{1}$$

where $u_0 = \omega a$ is the electron speed at $x = 0$,
and so

$$\bar{n} = \tfrac{1}{2}\pi n_0. \tag{2}$$

Half the electrons belong to the right-travelling, or the + stream, the others to the − stream. The charge density due to the undisturbed electrons, each with charge e, is

$$\begin{aligned} en(x) &= n_0 e(1-x^2/a^2)^{-\frac{1}{2}} \\ &= (2/\pi)\,\bar{n}e(1-x^2/a^2)^{-\frac{1}{2}} \end{aligned} \tag{3}$$

at position x. Let there be an equal and opposite background charge present, which remains unchanged in any disturbance. Consider what happens when the electrons are perturbed. In particular, let the electrons at position x in the $\pm$ streams be respectively displaced by distances $a\xi_{\pm}$ to the right. There will then be an unbalanced space charge $\tfrac{1}{2}n(x)\,ea(\xi_+ + \xi_-)$ to the right of position x, in first approximation. The electrostatic field at x becomes

$$E(x) = -2\pi n(x)\,ea(\xi_+ + \xi_-); \tag{4}$$

the equations of motion for the two streams become

$$d^2/dt^2\,(x + a\xi_{\pm}) + \omega^2(x + a\xi_{\pm}) + 2\pi n(x)\,a(e^2/m)\,(\xi_+ + \xi_-) = 0. \tag{5}$$

The undisturbed electrons would, however, obey the equation of motion

$$d^2x/dt^2 + \omega^2 x = 0. \tag{6}$$

After subtraction one can obtain the equations describing the disturbance and write them, in Eulerian form,

$$\left\{\frac{\partial}{\partial t} \pm u(x)\frac{\partial}{\partial x}\right\}^2 \xi_{\pm} + \omega^2\xi_{\pm} + 2\pi n(x)\,(e^2/m)\,(\xi_+ + \xi_-) = 0. \tag{7}$$

Convenient new independent variables can be introduced here by setting

$$x = a\sin\phi, \quad \tau = \omega t, \tag{8}$$

so that $n(x) = (2/\pi)\,\bar{n}\sec\phi$ and $u(x) = a\omega\cos\phi$;

further, let

$$\Lambda = 4\bar{n}e^2/m\omega^2. \tag{9}$$

The equations (7) become

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \phi}\right)^2 \xi_+ + \xi_+ + \tfrac{1}{2}\Lambda\,(\xi_+ + \xi_-)\sec\phi = 0, \tag{10}$$

for the + stream, and

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \phi}\right)^2 \xi_- + \xi_- + \tfrac{1}{2}\Lambda(\xi_+ + \xi_-)\sec\phi = 0, \tag{11}$$

for the − stream. When there is no τ-dependence the two equations apparently become identical; with $\Xi = \frac{1}{2}(\xi_+ + \xi_-)$ one obtains

$$\Xi'' + \Xi + \Lambda\Xi \sec\phi = 0. \tag{12}$$

We shall see later that this equation describes a stationary disturbance of small amplitude in an even mode, i.e. when Ξ is an even function of ϕ, but not in an odd mode.

The electric field at position ϕ is proportional to $\Xi \sec\phi$. It cannot become infinite anywhere, since the trap contains only a finite number of electrons, per unit area. Hence $\Xi \sec\phi$ should tend to a finite limit as ϕ tends to $\pm\frac{1}{2}\pi$, and so Ξ should vanish there. The solution for the steady state is really not quite so simple; the boundary conditions at $\pm\frac{1}{2}\pi$ have to be handled more carefully, since the linearization always breaks down there. We shall consider this problem in § 5.

3. Solutions of the eigenvalue problem

It is a straightforward eigenvalue problem to find solutions of (12), with the boundary conditions $\Xi = 0$ at $\phi = \pm\frac{1}{2}\pi$ (see, for example, Titchmarsh 1946, 1958). There is a discrete set of eigenvalues $\Lambda = \Lambda_0, \Lambda_1, \Lambda_2, \ldots$ for which appropriate solutions exist. The lowest Λ is, obviously, $\Lambda = 0$, and has the corresponding eigenfunction

$$\Xi_0 = 2^{-\frac{1}{2}} \cos\phi. \tag{13}$$

The factor $2^{-\frac{1}{2}}$ is introduced for purposes of normalization. The other Λ's are successively larger, and their corresponding eigenfunctions are alternately odd and even in ϕ. We define the sign of the Ξ's so as to make $d\Xi/d\phi$ negative when $\phi = \frac{1}{2}\pi$.

The orthogonality condition is derived by noting that

$$\Xi_r''\Xi_s - \Xi_r\Xi_s'' = -(\Lambda_r - \Lambda_s)\,\Xi_r\Xi_s \sec\phi. \tag{14}$$

On integration from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ the left-hand side vanishes, and so

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi_r\Xi_s \sec\phi\, d\phi = 0, \tag{15}$$

unless $\Lambda_r = \Lambda_s$, or $r = s$. We adopt the normalization condition

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi_r^2 \sec\phi\, d\phi = 1. \tag{16}$$

Putting $s = 0$ in (15) leads to

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi_r\, d\phi = 0, \tag{17}$$

unless $r = 0$. Finally, the Ξ_r's form a complete orthogonal set over the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

Numerical values for the Λ's and Ξ's can be readily estimated as follows. In the range $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$ expand

$$\cos\phi = \frac{2}{\pi}\left\{1 + 2\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{4r^2 - 1} \cos 2r\phi\right\}, \tag{18}$$

and, in the case of an odd solution, let

$$\Xi = \sum_{r=1}^{\infty} b_{2r} \sin 2r\phi. \tag{19}$$

The equation (12) can be written

$$(\Xi'' + \Xi) \cos\phi = -\Lambda\Xi. \tag{20}$$

After substitution from (18) and (19) both sides can be expressed as Fourier series in terms of $\sin 2n\phi$. This series is now cut off at some chosen N, and one can then equate coefficients. If $N = 3$ we find roots $\Lambda_1 \doteqdot 1{\cdot}76$, $\Lambda_3 \doteqdot 8{\cdot}2$, $\Lambda_5 \doteqdot 23{\cdot}8$; the value given for Λ_5 is not reliable in this approximation. The normalized eigenfunctions corresponding to Λ_1 and Λ_3 are

$$\left.\begin{aligned} \Xi_1 &\doteqdot 0{\cdot}60 \sin 2\phi - 0{\cdot}05 \sin 4\phi + 0{\cdot}01 \sin 6\phi, \\ \Xi_3 &\doteqdot 0{\cdot}23 \sin 2\phi + 0{\cdot}46 \sin 4\phi - 0{\cdot}10 \sin 6\phi. \end{aligned}\right\} \tag{21}$$

For the even eigenfunctions one uses the same expansion for $\cos\phi$ but writes

$$\Xi = \sum_{r=1}^{\infty} b_{2r-1} \cos(2r-1)\phi. \tag{22}$$

In any approximation $\Lambda = 0$ is a root. The next eigenvalue, in the approximation with $N = 4$, is $\Lambda_2 = 4{\cdot}12$; the first two even eigenfunctions are

$$\Xi_0 = 2^{-\frac{1}{2}} \cos\phi$$

and $$\Xi_2 \doteqdot 0{\cdot}20 \cos\phi + 0{\cdot}54 \cos 3\phi - 0{\cdot}10 \cos 5\phi + 0{\cdot}03 \cos 7\phi. \tag{23}$$

4. Stationary modes as limiting cases of time dependent disturbances

The time-dependent equations of a disturbance are

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\phi}\right)^2 \xi_+ + \xi_+ + \tfrac{1}{2}\Lambda(\xi_+ + \xi_-) \sec\phi = 0, \tag{24}$$

and $$\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\phi}\right)^2 \xi_- + \xi_- + \tfrac{1}{2}\Lambda(\xi_+ + \xi_-) \sec\phi = 0. \tag{25}$$

Write $$\tfrac{1}{2}(\xi_+ + \xi_-) = \Xi,$$

$$\tfrac{1}{2}(\xi_+ - \xi_-) = \Theta,$$

and let the time dependence of Ξ be like $\cosh\sigma\tau$. It is readily seen that Θ then has a time-dependence like $\sinh\sigma\tau$. Addition of (24) and (25) gives, with the time-dependence omitted,

$$\Xi'' + \Xi(1+\sigma^2) + \Lambda\Xi \sec\phi = -2\sigma\Theta', \tag{26}$$

and subtraction gives

$$\Theta'' + \Theta(1+\sigma^2) = -2\sigma\Xi'. \tag{27}$$

Consider now the solution of (26) and (27), when σ is small. One would try then to approximate (27) by

$$\Theta'' + \Theta = -2\sigma\Xi'. \tag{28}$$

The boundary conditions on Θ are that $\Theta = 0$ at $\phi = \pm\frac{1}{2}\pi$, or else $\xi_+\{\equiv \frac{1}{2}(\Xi+\Theta)\}$ cannot go over smoothly there into $\xi_-\{\equiv \frac{1}{2}(\Xi-\Theta)\}$. The solution of (28) which vanishes at $\phi = -\frac{1}{2}\pi$ is

$$\Theta(\phi) = -2\sigma\int_{-\frac{1}{2}\pi}^{\phi} \Xi'(u)\sin(\phi-u)\,du, \tag{29}$$

and so

$$\Theta(\tfrac{1}{2}\pi) = -2\sigma\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi'(u)\cos u\,du. \tag{30}$$

This also vanishes when Ξ is an even function, and so a solution exists in this approximation. Θ is then of order σ, and goes to zero in the limit as $\sigma \to 0$. Thus equation (12) correctly describes an even stationary mode of small amplitude.

But for an odd mode one cannot in general find a solution of (28) to satisfy the boundary conditions. This equation is, therefore, a poor approximation. Returning to the full equation (29) we see that in this case a possible solution is approximately

$$\Theta = -\frac{2\sigma}{1+\frac{1}{2}\sigma^2}\int_0^{\phi} \Xi'(u)\sin\{(1+\tfrac{1}{2}\sigma^2)(\phi-u)\}\,du + A\cos(1+\tfrac{1}{2}\sigma^2)\,\phi$$

$$\doteqdot -2\sigma\int_0^{\phi} \Xi'(u)\sin(\phi-u)\,du + A\cos\phi - \tfrac{1}{2}A\sigma^2\phi\sin\phi. \tag{31}$$

Θ vanishes at $\phi = \pm\frac{1}{2}\pi$ if

$$A = -\frac{4}{\pi\sigma}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi'(u)\cos u\,du \equiv -\frac{4}{\pi\sigma}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi(u)\sin u\,du. \tag{32}$$

The dominant part of the expression for Θ is now

$$\Theta^{(m)} = -\frac{4\cos\phi}{\pi\sigma}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi(u)\sin u\,du. \tag{33}$$

On substitution into (26) one finds that

$$\Xi'' + \Xi + \Lambda\Xi\sec\phi = -\frac{8}{\pi}\sin\phi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi(u)\sin u\,du, \tag{34}$$

having neglected the contribution $\sigma^2\Xi$ on the left-hand side. The equation is valid if σ is small enough, and so, as σ tends to zero. It is the appropriate equation for finding the stationary odd modes of the system. The boundary conditions are that $\Xi = 0$ at $\phi = \pm\frac{1}{2}\pi$.

But in the limit as σ tends to zero, relation (33) suggests that $\Theta^{(m)}$ becomes infinite. On inserting the assumed time dependence one sees that this is not a real infinity, but that the dominant part of Θ becomes

$$\Theta^{(m)}\sinh\sigma\tau \approx -\left(\frac{4}{\pi}\right)\tau\cos\phi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi(u)\sin u\,du$$

$$\equiv \tau\,\delta\omega\cos\phi, \quad \text{say.} \tag{35}$$

This term grows linearly with time, but is never infinite. It can be removed by suitably re-defining the motion of the reference particles, as follows.

An undisturbed electron, in the right-moving stream, say, has its angular co-ordinate given by $\phi = \tau + \epsilon$, where ϵ is a phase angle. Its position is then given by

$$x = a \sin\phi = a \sin(\tau + \epsilon). \tag{36}$$

If only a disturbance of the form (35) is present the electron is displaced to

$$x + \delta x = a(\sin\phi + \tau\delta\omega\cos\phi) \doteqdot a\sin(\phi + \tau\delta\omega) = a\sin\{\tau(1+\delta\omega)+\epsilon\}. \tag{37}$$

The effect of the $\Theta^{(m)}$ term is to change the angular frequency of the electrons in the trap. The $\Theta^{(m)}$ term can, therefore, be simply removed by giving a slightly different definition to the motion of the reference particles. Such a change produces no space charge anywhere, and therefore leads to no physical effects. One readily sees why the frequency of the electrons has to be changed like this, for the trapping field is odd, and if the disturbance field is odd as well, then they add together coherently. This does not happen for an even disturbance field.

We go on to find the values of Λ such that solutions of (34) can be found to satisfy the boundary conditions. Expand

$$\Xi = \sum_{r=1}^{\infty} c_{2r-1}\Xi_{2r-1}. \tag{38}$$

On substitution into (34) one then obtains that

$$\sec\phi \sum_{r=1}^{\infty} c_{2r-1}(\Lambda - \Lambda_{2r-1})\,\Xi_{2r-1} = -4\sum_{r=1}^{\infty} B_{2r-1}c_{2r-1}\sin\phi, \tag{39}$$

where

$$B_{2r-1} \equiv \frac{2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi_{2r-1}(u)\sin u\,du. \tag{40}$$

Multiply both sides of (39) by Ξ_{2s-1} and integrate over ϕ to find that

$$c_{2s-1}(\Lambda - \Lambda_{2s-1}) = -2\pi B_{2s-1}\sum_{r=1}^{\infty} c_{2r-1}B_{2r-1}. \tag{41}$$

Now multiply both sides of (43) by $B_{2s-1}/(\Lambda - \Lambda_{2s-1})$ and sum over all s to find that

$$1 + 2\pi\sum_{s=1}^{\infty}\frac{B_{2s-1}^2}{\Lambda - \Lambda_{2s-1}} = 0. \tag{42}$$

The coefficients B_{2s-1} may be expressed in another way. From the equation defining Ξ_{2s-1} one has that

$$\begin{aligned}\Lambda_{2s-1}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Xi_{2s-1}\sin\phi\,d\phi &= -\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\Xi_{2s-1}'' + \Xi_{2s-1})\sin\phi\cos\phi\,d\phi \\ &= \frac{3}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Xi_{2s-1}\sin 2\phi\,d\phi = \frac{3\pi}{4}b_2^{(2s-1)},\end{aligned} \tag{43}$$

after some integration by parts, and on using expansion (19). The superfix $2s-1$ now indicates that the expansion refers to the function Ξ_{2s-1}. Thus

$$B_{2s-1} \equiv (3/2\Lambda_{2s-1})\,b_2^{(2s-1)}. \tag{44}$$

Values for $b_2^{(1)}$ and $b_2^{(3)}$ can be read off from relation (21); other b_2 coefficients will not be needed in the approximate calculation which follows. But, in general, values of Λ which allow the existence of a stationary odd mode can be found, i.e.

$$F(\Lambda) \equiv 1 + \frac{9\pi}{2} \sum_{s=1}^{\infty} \frac{\{b_2^{(2s-1)}\}^2}{\Lambda_{2s-1}^2(\Lambda - \Lambda_{2s-1})} = 0. \tag{45}$$

This equation in Λ has a countable infinity of non-negative roots. Clearly $F(\Lambda) \to \mp\infty$ whenever $\Lambda \to \Lambda_{2s-1} \mp 0$, respectively. There is then just one root between each Λ_{2s-1} and Λ_{2s+1}.

Note that $\Xi = \phi \cos\phi$ satisfies the boundary conditions and the equation (34), when $\Lambda = 0$. Hence $F(0) = 0$. Further, $F(\Lambda)$ is a strictly decreasing function of Λ when Λ is negative. Thus $\Lambda = 0$ is the smallest root.

The next higher root of $F(\Lambda) = 0$ lies between Λ_1 and Λ_3. On insertion of numerical values one finds that it is actually quite close to Λ_3. Only the coefficients $b_2^{(1)}$ and $b_2^{(3)}$ make an appreciable contribution and need to be retained in the calculation. On setting $\Lambda = \Lambda_3 + \delta\Lambda$ and neglecting $\delta\Lambda$ in comparison with Λ one finds that

$$1 + \frac{9\pi}{2}\left[\frac{\{b_2^{(1)}\}^2}{\Lambda_1^2(\Lambda_3 - \Lambda_1)} + \frac{\{b_2^{(3)}\}^2}{\Lambda_3^2\,\delta\Lambda}\right] \doteqdot 0, \tag{46}$$

so that $\delta\Lambda \doteqdot -0{\cdot}0088$. The displacement of Λ from Λ_3 is only very small. For higher odd modes it is even less significant.

In summary then, even stationary disturbances of small amplitude can occur near $\Lambda = \Lambda_{2s}$ but the odd modes are displaced from Λ_{2s-1} to Λ_{2s-1}^*. The lowest of these is $\Lambda_1^* = 0$. The other Λ^* are only slightly smaller than the corresponding Λ.

5. Non-linear effects at the boundary

Only linearized equations have been used so far to describe the plasma waves. But the linearization breaks down near $\phi = \pm\frac{1}{2}\pi$, and a different solution has to be found there.

Consider the motion near $\phi = \frac{1}{2}\pi$ for an even time-dependent disturbance. Let the motion be so defined that an electron is furthest in the direction of x increasing (or furthest to the right) at the instant when its reference particle passes through $\phi = \frac{1}{2}\pi$. This can always be arranged, if necessary, by adding an amount $\pm\theta_* \cos\phi$ to $\xi_\pm$. The alteration does not change the value of $\frac{1}{2}(\xi_+ + \xi_-)$, or of its derivative. Let $\xi(\frac{1}{2}\pi) = \xi_*$, and write $\psi \equiv \frac{1}{2}\pi - \phi$.

Two cases arise, depending on whether ξ_* is negative or positive. If ξ_* is negative, each electron remains to the left of its reference particle while in the immediate vicinity of $\psi = 0$. Consider then the charge to the right of an electron, whose reference particle is at position ψ. Since the reference particles are uniformly distributed with respect to ψ, the total charge due to the electrons in this region is $n_0 ea\psi$. If the reference particle is at ψ, the electron is actually located at

$$x = a(\cos\psi + \xi) = a\cos\psi'. \tag{47}$$

Hence the background charge to the right of that electron contributes $-n_0 ea\psi'$, and the net charge becomes $n_0 ea(\psi-\psi')$; the corresponding electric field is $-4\pi n_0 ea(\psi-\psi')$. The equation of motion becomes

$$\omega^2 \frac{d^2\xi}{d\psi^2}+\omega^2\xi+\frac{4\pi n_0 e^2}{m}(\psi-\psi') = 0,$$

or

$$(d^2\xi/d\psi^2)+\xi+\Lambda(\psi-\psi') = 0. \tag{48}$$

Since ψ and ψ' are small, one can now approximate by setting $\cos\psi \doteqdot 1-\frac{1}{2}\psi^2$ and $\cos\psi' \doteqdot 1-\frac{1}{2}\psi'^2$, so that

$$\psi' = (\psi^2-2\xi)^{\frac{1}{2}}, \tag{49}$$

and (48) becomes

$$(d^2\xi/d\psi^2)+\xi+\Lambda[\psi-(\psi^2-2\xi)^{\frac{1}{2}}] = 0. \tag{50}$$

When ψ^2 much exceeds $2|\xi|$ one can expand under the square root sign, and recover the form of the linearized equation appropriate near $\psi = 0$. But there is a region within which this cannot be done. Here one may approximate by setting $\xi \doteqdot -|\xi_*|$ and integrate, with the boundary condition $d\xi/d\psi = 0$ at $\psi = 0$, to find that

$$d\xi/d\psi = \psi|\xi_*|-\tfrac{1}{2}\Lambda[\psi^2-\psi(\psi^2+2|\xi_*|)^{\frac{1}{2}}]+\Lambda|\xi_*|\sinh^{-1}\psi/(2|\xi_*|)^{\frac{1}{2}}. \tag{51}$$

In the region where ψ^2 much exceeds $2|\xi_*|$ this leads to

$$d\xi/d\psi \approx \Lambda|\xi_*|\log 2\psi-\tfrac{1}{2}\Lambda|\xi_*|\log 2|\xi_*| \approx \Lambda|\xi_*|\log\psi-\tfrac{1}{2}\Lambda|\xi_*|\log|\xi_*|, \tag{52}$$

and here the second term on the right dominates. We also see that $d\xi/d\psi$ is of order $\xi_*\log|\xi_*|$, and so the change in the value of ξ over a range ψ is at most of order $\psi|\xi_*|\log|\xi_*|$. One can therefore consistently neglect the variation of ξ in the boundary domain, as long as $\psi\log|\xi_*|$ is kept small. The linearized solution, which is valid outside the boundary domain, must therefore be matched smoothly to the conditions

$$\xi = -|\xi_*|, \quad \frac{d\xi}{d\phi}\left(\equiv -\frac{d\xi}{d\psi}\right) = \tfrac{1}{2}\Lambda|\xi_*|\log|\xi_*|. \tag{53}$$

But if ξ_* is positive, the equation (50) cannot be used all the way to $\psi = 0$, since $\psi^2-2\xi$ eventually becomes negative, and then all the background charge is to the left of the electrons. The electric field on a given electron can then be found simply in terms of those electrons which are to its right, so that

$$(d^2\xi/d\psi^2)+\xi+\Lambda\psi = 0 \tag{54}$$

replaces (50) there. The calculation can be carried through as before, but now in two parts. One finds that the required boundary conditions to be satisfied at the edge of the linear domain are that

$$\xi = \xi_*, \quad \frac{d\xi}{d\phi}\left(\equiv -\frac{d\xi}{d\psi}\right) = -\tfrac{1}{2}\Lambda\xi_*\log|\xi_*|, \tag{55}$$

when $\phi = \frac{1}{2}\pi$. Except for small Λ the ratio $|\xi|:|d\xi/d\phi|$ is small at the boundary. Hence an adequate boundary condition for a small amplitude disturbance is $\xi = 0$ when $\phi = \frac{1}{2}\pi$. Stationary solutions in the neighbourhood of $\Lambda = 0$ have to be discussed separately (see §7).

In discussing the boundary layer for an odd mode one finds that the only difference is that there is now the additional term

$$\frac{8}{\pi}\sin\phi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Xi\sin u\,du$$

to be added to equation (50). Near the boundary $\sin\phi \doteqdot 1$; further the integral is evidently of the same order as $\Xi'(\frac{1}{2}\pi)$. On following through the calculation one obtains estimates for ξ which are the same as in (53) or (55), to the accuracy required. The value for $d\xi/d\phi$ is changed by an additional term of order $\psi\Xi'(\frac{1}{2}\pi)$. This change is negligible, since $\Xi'(\frac{1}{2}\pi)$ is of the same order as $d\xi/d\phi$, while ψ is small. Relations (53) and (55) therefore apply to even and to odd modes.

6. Unstable disturbances

It will now be shown that, as one might expect, a new mode becomes unstable whenever Λ increases through a positive Λ_{2s} or Λ^*_{2s-1}.

Consider first an even mode. Let $\Lambda = \Lambda_{2s}+\lambda$; the time-dependent equations for Θ and Ξ may then be written

$$L_{2s}(\Xi) \equiv \Xi''+\Xi+\Lambda_{2s}\Xi\sec\phi = -\sigma^2\Xi_{2s}-2\sigma\Theta'_{2s}-\lambda\Xi_{2s}\sec\phi, \tag{56}$$

and

$$\Theta''_{2s}+\Theta_{2s} = -2\sigma\Xi'_{2s}, \tag{57}$$

in good enough approximation. If the solution is expanded in the form

$$\Xi = \sum_{r=0}^{\infty}\alpha_{2r}\Xi_{2r}, \tag{58}$$

then the boundary conditions on Ξ at $\phi = \pm\frac{1}{2}\pi$ are automatically satisfied. Further (56) becomes

$$\sec\phi\sum_{r=0}^{\infty}(\Lambda_{2s}-\Lambda_{2r})\,\alpha_{2r}\Xi_{2r} = -(\sigma^2\Xi_{2s}+2\sigma\Theta'_{2s}+\lambda\Xi_{2s}\sec\phi). \tag{59}$$

The coefficient of Ξ_{2s} on the left-hand side vanishes. On multiplying through by Ξ_{2s} and integrating both sides, one finds that the relation between λ and σ is

$$\sigma^2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Xi^2_{2s}\,d\phi+2\sigma\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Xi_{2s}\Theta'_{2s}\,d\phi = -\lambda. \tag{60}$$

We shall now show that

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Xi^2_{2s}\,d\phi+\frac{2}{\sigma}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Xi_{2s}\Theta'_{2s}\,d\phi < 0, \tag{61}$$

so that real positive values of σ occur when λ is positive, and the waves are then unstable.

To establish the result expand Θ_{2s}, which is an odd function of ϕ,

$$\Theta_{2s} = \sigma\sum_{r=1}^{\infty}\beta_{2r}\sin 2r\phi. \tag{62}$$

Hence from (64)

$$\Xi'_{2s} = \frac{1}{2}\sum_{r=1}^{\infty}(4r^2-1)\,\beta_{2r}\sin 2r\phi,$$

and
$$\Xi_{2s} = -\frac{1}{2}\sum_{r=1}^{\infty}\frac{4r^2-1}{2r}\beta_{2r}\cos 2r\phi + A; \tag{63}$$

the constant of integration A must vanish since we know that $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Xi_{2s}\,d\phi = 0$. Hence

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left\{\Xi_{2s}^2 + \frac{2}{\sigma}\Xi_{2s}\Theta'_{2s}\right\}d\phi = \frac{\pi}{2}\sum_{r=1}^{\infty}\left\{\left(\frac{4r^2-1}{4r}\right)^2\beta_{2r}^2 - (4r^2-1)\,\beta_{2r}^2\right\}; \tag{64}$$

since this is clearly negative, our result is established for the even modes.

For an odd mode we must first separate out the main part $\Theta^{(m)}_{2s-1}$ of Θ_{2s-1}, so that

$$\Theta_{2s-1} = \Theta^{(m)}_{2s-1} + \theta_{2s-1}, \tag{65}$$

and θ_{2s-1} is orthogonal to $\cos\phi$. The equations for a slightly unstable wave become

$$L^*_{2s-1}(\Xi) \equiv \Xi'' + \Xi + \Lambda^*_{2s-1}\Xi\sec\phi + \frac{8}{\pi}\sin\phi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Xi\sin u\,du$$

$$= -\sigma^2\Xi^*_{2s-1} - 2\sigma\theta_{2s-1} - \lambda\Xi^*_{2s-1}\sec\phi \tag{66}$$

and
$$\theta''_{2s-1} + \theta_{2s-1} = -2\sigma\{\Xi^{*\prime}_{2s-1} - B^*_1\cos\phi\}. \tag{67}$$

The coefficient B^*_1 is defined in the expansion

$$\Xi^*_{2s-1} = \sum_{r=1}^{\infty} B^*_{2r-1}\sin(2r-1)\,\phi. \tag{68}$$

Once again unstable waves occur for positive values of λ if

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\{\Xi^{*2}_{2s-1} + (2/\sigma)\,\Xi^*_{2s-1}\theta'_{2s-1}\}\,d\phi < 0, \tag{69}$$

or if
$$\sum_{r=1}^{\infty} B^{*2}_{2r-1} - 4\sum_{r=2}^{\infty}\frac{(2r-1)^2}{(2r-1)^2-1}B^{*2}_{2r-1} < 0. \tag{70}$$

The requirement is met if it can be shown that

$$B^{*2}_1 < 3\sum_{r=2}^{\infty} B^{*2}_{2r-1}, \tag{71}$$

and this is readily established as follows. An alternative expansion for Ξ^*_{2s-1} is

$$\Xi^*_{2s-1} = \sum_{r=1}^{\infty} b^*_{2r}\sin 2r\phi. \tag{72}$$

Evidently
$$\sum_{r=1}^{\infty} B^{*2}_{2r-1} = \sum_{r=1}^{\infty} b^{*2}_{2r}, \tag{73}$$

and so (71) is proved if it can be shown that

$$B^{*2}_1 < 3\left[\sum_{r=1}^{\infty} b^{*2}_{2r} - B^{*2}_1\right],$$

or
$$B^{*2}_1 < \frac{3}{4}\sum_{r=1}^{\infty} b^{*2}_{2r},$$

or, even better, that $$B_1^{*2} < \tfrac{3}{4} b_2^{*2}. \tag{74}$$
To do so we note that

$$0 = \frac{1}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin 2\phi \, L_{2s-1}^*(\Xi_{2s-1}^*)\, d\phi$$
$$= -\frac{3}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi_{2s-1}^* \sin 2\phi \, d\phi + \Lambda_{2s-1}^* \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi_{2s-1}^* \sin\phi \, d\phi,$$

so that $$\tfrac{3}{2} b_2^* = \Lambda_{2s-1}^* B_1^*. \tag{75}$$

Now Λ_{2s-1}^* certainly exceeds $\surd 3$ when s exceeds unity. The result therefore follows, except in the case that $\Lambda = \Lambda_1^* = 0$.

7. Non-linear effects on stationary disturbances

We come now to consider the effect of the non-linearities which arise near $\phi = \pm\frac{1}{2}\pi$ when the wave has a finite amplitude. The boundary condition (55) can then be written in the form

$$\Xi(\phi) \approx -\frac{2\Xi'(\phi)}{\Lambda \log|\xi_*|} \tag{76}$$

as $\phi \to \frac{1}{2}\pi$, and in an analogous form near $\phi = -\frac{1}{2}\pi$. We deal with the even modes first.

To take account of the changed boundary condition we re-write the equation for Ξ in the form

$$\Xi'' + \Xi + \Lambda\Xi \sec\phi = -\frac{2\Xi'(\frac{1}{2}\pi)}{\Lambda \log|\xi_*|}\,\delta'(\tfrac{1}{2}\pi - \phi) \tag{77}$$

and so $\Xi \to -2\Xi'(\frac{1}{2}\pi)/(\Lambda \log|\xi_*|)$ as $\phi \to \frac{1}{2}\pi - 0$. If Λ is only little altered by the change, then one may write $\Lambda = \Lambda_{2s} + \lambda$, and again regard λ as small. On using the expansion $\Xi = \sum_{r=0}^{\infty} \alpha_{2r}\Xi_{2r}$ one finds that

$$\sec\phi \sum_{r=0}^{\infty} \alpha_{2r}(\Lambda_{2s} - \Lambda_{2r})\,\Xi_{2r} = -\frac{2\Xi_{2s}'(\frac{1}{2}\pi)}{\Lambda_{2s}\log|\xi_*|}\,\delta'(\tfrac{1}{2}\pi - \phi) - \lambda\Xi_{2s}\sec\phi. \tag{78}$$

Multiply through by Ξ_{2s} and integrate from 0 to $\frac{1}{2}\pi$ to find that

$$\tfrac{1}{2}\lambda = -\frac{2\Xi_{2s}'(\frac{1}{2}\pi)}{\Lambda_{2s}\log|\xi_*|}\int_0^{\frac{1}{2}\pi} \Xi_{2s}(\phi)\,\delta'(\tfrac{1}{2}\pi - \phi)\,d\phi$$

or $$\lambda = -4\Xi_{2s}'^{2}(\tfrac{1}{2}\pi)/(\Lambda_{2s}\log|\xi_*|); \tag{79}$$

and this is positive, since $\log|\xi_*|$ is negative. The amplitude of the disturbance goes to zero with ξ_*, and so does λ.

The argument for the odd modes is entirely similar, and we find that the non-linear boundary conditions produce a change in Λ^* amounting to

$$\lambda = -4\{\Xi_{2s-1}^{*\prime}(\tfrac{1}{2}\pi)\}^2/(\Lambda_{2s-1}^*\log|\xi_*|), \tag{80}$$

provided $s \geqslant 2$.

The non-linear corrections take on a somewhat different form when the electron density is low, and so Λ is small. Consider therefore a small value of

Λ, say $\Lambda = \lambda_0$. For this parameter the linearized equation describing a stationary disturbance in the even mode is

$$\Xi'' + \Xi + \lambda_0 \Xi \sec\phi = 0, \tag{81}$$

and has the solution

$$\Xi = A(\cos\phi - \lambda_0), \tag{82}$$

correct to the first order in λ_0. This solution matches on smoothly to the boundary layer provided that

$$\Xi(\tfrac{1}{2}\pi) = \xi_*, \quad \text{and} \quad \Xi'(\tfrac{1}{2}\pi) = -\tfrac{1}{2}\lambda_0 \xi_* \log|\xi_*|. \tag{83}$$

This means that we must have

$$\xi_* = -\lambda_0 A \quad \text{and} \quad -A = -\tfrac{1}{2}\lambda_0 \xi_* \log|\xi_*|, \tag{84}$$

so that the solution is either trivial ($\xi_* = 0$) or else

$$\lambda_0^2 = -2/\log|\xi_*|. \tag{85}$$

Hence

$$\xi_* = \pm \exp(-2/\lambda_0^2) \tag{86}$$

and

$$\Xi = \mp (1/\lambda_0)(\cos\phi - \lambda_0)\exp(-2/\lambda_0^2). \tag{87}$$

To be specific, take the disturbance with the upper sign, for which the electric field, in dimensionless form, is

$$\mathscr{E} = -\lambda_0 \Xi \sec\phi = \exp(-2/\lambda_0^2)(1 - \lambda_0 \sec\phi). \tag{88}$$

The diagrams in figure 1 show plots of $\mathscr{E}$ and Ξ against $\sin\phi$, the distance from the centre of the trap. In this mode there is a potential difference

$$\Delta v = -2e^{-2/\lambda_0^2}(1 - \tfrac{1}{2}\lambda_0 \pi)$$

between the ends of the trap. It is also seen that the electrons overshoot the background plasma on the right-hand side, and 'undershoot' it on the left.

The linearized equation describing a stationary odd disturbance is

$$\Xi'' + \Xi + \frac{8}{\pi}\sin\phi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi \sin\phi \, d\phi = -\lambda_0 \Xi \sec\phi, \tag{89}$$

and once again λ_0 is supposed to be small. When λ_0 vanishes, (89) has the solution

$$\Xi = B\phi\cos\phi; \tag{90}$$

to the first order in λ_0 equation (89) therefore becomes

$$\Xi'' + \Xi + \frac{8}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi \sin\phi \, d\phi = -\lambda_0 B\phi. \tag{91}$$

It is easily verified that the solution required here is

$$\Xi = B\phi\cos\phi + B\lambda_0\{(4/\pi)\sin\phi - \phi\}. \tag{92}$$

When $\phi = \frac{1}{2}\pi$, then,

$$\left.\begin{aligned} \Xi &= -B\lambda_0(\tfrac{1}{2}\pi - 4/\pi) = \xi_* \\ \text{and} \quad d\Xi/d\phi &= -\tfrac{1}{2}B\pi = -\tfrac{1}{2}\lambda_0\xi_*\log|\xi_*|, \end{aligned}\right\} \tag{93}$$

to sufficient accuracy. On elimination of B one finds that

$$\lambda_0^2 = -2\{(1-8/\pi^2)\log|\xi_*|\}^{-1} \tag{94}$$

or

$$|\xi_*| = \exp\left\{-\frac{2}{1-8/\pi^2}\frac{1}{\lambda_0^2}\right\}. \tag{95}$$

Going back to (93) one now finds that

$$B = \pm\frac{2}{\pi\lambda_0(1-8/\pi^2)}\exp\left\{-\frac{2}{(1-8/\pi^2)\lambda_0^2}\right\}. \tag{96}$$

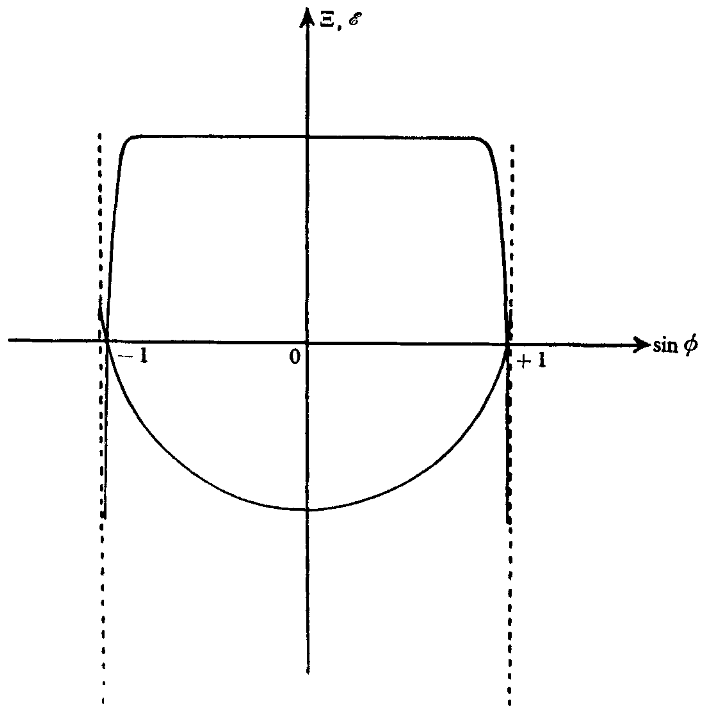

FIGURE 1. The semi-circular curve shows how the displacement Ξ of an electron varies with distance from the centre of the trap, in the lowest even stationary mode. The other curve shows the variation of the electric field ℰ. Notice how both ℰ and Ξ reverse signs near the ends of the trap.

These equations describe a stationary odd disturbance which can exist in the plasma even for small λ_0. This time the amplitude is much smaller, for a given λ_0, than in an even disturbance, because of the presence of the factor $(1-8/\pi^2) \doteqdot 0{\cdot}19$ in the denominator of the exponent.

There is no potential difference between the ends of the trap in an odd mode, but the potential difference between the centre and the edge of the trap is, in dimensionless terms,

$$= -B\int_0^{\frac{1}{2}\pi}[\phi\cos\phi+\lambda_0\{(4/\pi)\sin\phi-\phi\}]\cos\phi\,d\phi \doteqdot -B(0{\cdot}37-0{\cdot}06\lambda_0). \tag{97}$$

8. Stability of the plasma when the electron density is low

We shall consider the stability of disturbances in plasmas for which Λ is close to zero. For larger values of Λ, say when Λ was close to Λ_{2n}, the procedure was to set $\Lambda = \Lambda_{2n} + \lambda$ and to consider a disturbance of infinitesimal amplitude, with a time dependence of the form $\cosh \sigma\tau$. On applying a perturbation procedure to the linearized equations we then found a linear relation between σ^2 and λ. Solutions in regions with σ^2 positive were judged to be unstable. But this method cannot be used near $\Lambda = 0$, for the coefficient of σ^2 vanishes there, as

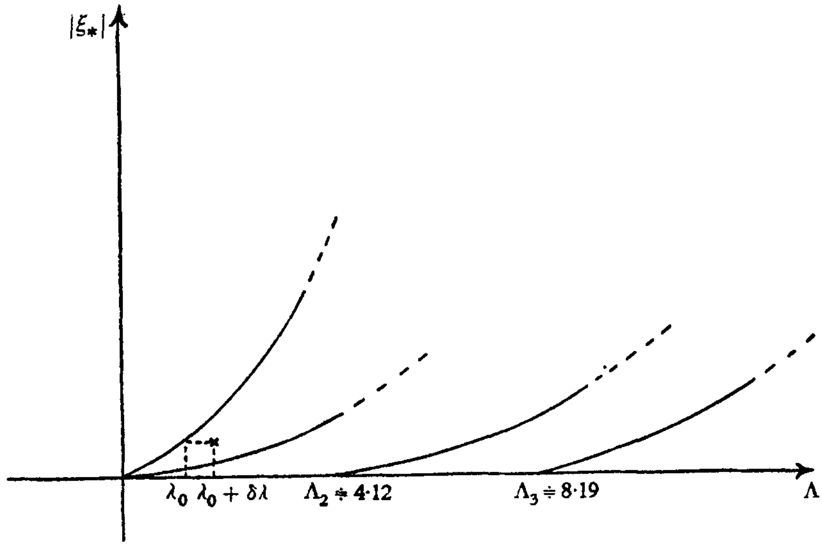

FIGURE 2. Curves relating values of $|\xi_*|$ and Λ for which stationary disturbances can exist. Reading from the left the curves refer to the lowest even, the lowest odd, the second even and the second odd modes. A particular mode is unstable when $(|\xi_*|, \Lambda)$ lies below the corresponding stationary solution curve. The points $(|\xi_*|, \lambda_0)$ and $(|\xi_*|, \lambda_0 + \delta\lambda)$ are shown to illustrate the argument leading to a stability criterion for the lowest even mode (see § 8).

is easily verified. Hence we shall look at the time-dependence of a solution of finite amplitude, whose representative point lies just below the curve for stationary solutions in the (ξ_*, Λ)-plane.

Let therefore $X_0(\phi)$ be the stationary even solution when $\Lambda = \lambda_0$, and let $X_0(\frac{1}{2}\pi) = \xi_*$, so that $X_0'(\frac{1}{2}\pi) = -\frac{1}{2}\lambda_0 \xi_* \log |\xi_*|$. To a good enough approximation then

$$X_0 = A(\cos\phi - \lambda_0), \tag{98}$$

where

$$A = \pm \lambda_0^{-1} e^{-2/\lambda_0^2} \quad \text{and} \quad X'(\tfrac{1}{2}\pi) = -A.$$

Now keep ξ_* fixed but consider the solution for $\Lambda = \lambda_0 + \delta\lambda$. If $\delta\lambda$ is positive, the point $(\xi_*, \lambda_0 + \delta\lambda)$ lies below the stationary solution curve. The corresponding disturbance is therefore time-dependent. Let it be $\Xi(\phi) \cosh \sigma\tau$.

To satisfy the boundary conditions now requires that

$$\begin{aligned} \Xi'(\tfrac{1}{2}\pi) &= -\tfrac{1}{2}(\lambda_0 + \delta\lambda)\, \xi_* \log |\xi_*| \\ &= (1 + \delta\lambda/\lambda_0)\, X_0'(\tfrac{1}{2}\pi). \end{aligned} \tag{99}$$

As in previous calculations

$$\Theta'' + \Theta = -2\sigma X_0' = 2\sigma A \sin\phi. \tag{100}$$

The solution for which $\Theta(\frac{1}{2}\pi) = 0$ is

$$\Theta = -\sigma A \phi \cos\phi, \tag{101}$$

so that the perturbed equation which Ξ must satisfy is

$$\Xi'' + \Xi + \lambda_0 \Xi \sec\phi = -\sigma^2 X_0 - 2\sigma\Theta' - \delta\lambda X_0 \sec\phi \equiv H(\phi), \quad \text{say}. \tag{102}$$

Multiply both sides of this equation by X_0 and integrate from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. After some integration by parts, and on using the fact that X_0 satisfies

$$X_0'' + X_0 + \lambda_0 X \sec\phi = 0,$$

one finds that

$$[X_0(\phi)\,\Xi'(\phi) - X_0'(\phi)\,\Xi(\phi)]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} H(\phi)\,X_0(\phi)\,d\phi. \tag{103}$$

Now $$X_0(\tfrac{1}{2}\pi) = \Xi(\tfrac{1}{2}\pi) = \xi_* \doteqdot -\lambda_0 A$$

and $$\Xi'(\tfrac{1}{2}\pi) - X_0'(\tfrac{1}{2}\pi) = -\tfrac{1}{2}\delta\lambda\xi_* \log|\xi_*| \doteqdot -(\delta\lambda/\lambda_0)\,A.$$

The left-hand side of (103) therefore becomes $2A^2\delta\lambda$. On substitution from (98) and (99) into the right-hand side of (103) we then find, after some reduction, that

$$2A^2\delta\lambda = 4A^2\lambda_0\sigma^2 - 2A^2\delta\lambda, \tag{104}$$

to a good enough approximation. Thus

$$\sigma^2 = \delta\lambda/\lambda_0, \tag{105}$$

so that σ^2 is positive, and the disturbance unstable, if $\delta\lambda$ is positive. This means that once again points below the steady solution curve represent unstable disturbances.

We now consider odd disturbances, and the method is quite similar. Let $X_0^*(\phi)$ be the stationary odd disturbance when $\Lambda = \lambda_0$. Once again

$$X_0^*(\tfrac{1}{2}\pi) = \xi_* \doteqdot -\lambda_0 B(\tfrac{1}{2}\pi - 4/\pi)$$

and $$(dX_0^*/d\phi)_{\frac{1}{2}\pi} = -\tfrac{1}{2}\lambda_0\xi_* \log|\xi_*| \doteqdot -\tfrac{1}{2}\pi B.$$

Further, $$X_0^*(\phi) \doteqdot B\phi\cos\phi + B\lambda_0\{(4/\pi)\sin\phi - \phi\}.$$

As usual we split up $$\Theta = \Theta^{(m)} + \theta,$$

where $\Theta^{(m)}$ contains the $\cos\phi$-dependent part. Θ now satisfies the equation

$$\theta'' + \theta = -\sigma^2\Theta^{(m)} - 2\sigma X_0^{*\prime},$$

and after some computation one finds that the approximate solution is

$$\theta = -\tfrac{1}{2}B\sigma(\phi^2\cos\phi - q\cos\phi) - 2\sigma B\lambda_0\left(\frac{2}{\pi}\phi\sin\phi + \frac{3}{\pi}\cos\phi - 1\right). \tag{106}$$

The equation for Ξ becomes

$$\Xi'' + \Xi + \lambda_0\Xi\sec\phi + \frac{8}{\pi}\sin\phi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Xi\sin\phi\,d\phi$$
$$= -\sigma^2 X_0^* - 2\sigma\theta' - \delta\lambda X_0^*\sec\phi. \tag{107}$$

Once again we multiply through by $X_0^*(\phi)$ and integrate from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. The resulting relation is that

$$\sigma^2 = \frac{3(\pi^2-8)}{5\pi^2-48}\frac{\delta\lambda}{\lambda_0} \doteqdot 3{\cdot}8\frac{\delta\lambda}{\lambda_0}, \tag{108}$$

and again the points below the steady-solution curve represent unstable disturbances.

9. Discussion

In many ways the properties of the plasma in the trap resemble those of the uniform double-streaming plasma. Both can carry unstable space-charge waves, and both have certain critical values for the electron density, above which new unstable waves can switch on.

But there are also some significant differences. The trapping field eventually stops the electrons in the + stream and turns them into the − stream. Since there can be no infinitely large forces anywhere the transition must be smooth, and this affects the boundary conditions at the end of the trap. Non-linear equations must be used to describe the boundary regions and they show that a layer of space charge will always build up there. They also permit the calculation of the amplitude of a stationary disturbance which the plasma can carry, in a given mode, at a mean density $\bar{n}$ somewhat larger than the critical value for that mode. Disturbances of smaller amplitude will be unstable at mean density $\bar{n}$, and will grow to reach the amplitude of the corresponding stationary wave.

Again, in the double-streaming plasma, an electron comes from $-\infty$ say, passes once through the disturbed region and then on to ∞. But in the trapped plasma a given electron passes repeatedly through the disturbed region. In an odd mode the effect of the disturbance field on an electron can add coherently to that of the trapping field, and in the lowest modes a large phase difference may then be built up between an electron and its reference particle. As a result stationary odd modes are described by equations different from those for the even modes, and the critical values of the electron densities are therefore displaced for the odd modes. This is particularly marked in the case of the lowest odd mode, whose critical value is displaced to zero.

Finally at any given small electron density the trapped plasma can carry an even or an odd wave of finite amplitude, and once again waves of smaller amplitude are unstable for that density in that mode. A minimum density is needed before instability sets in at a given wavelength in the double-streaming plasma.

For simplicity's sake the argument so far has been restricted to a mono-energetic electron plasma in a one-dimensional potential well, but one might ask which of the conclusions remains valid if some of the restrictions are relaxed. For example, the non-linear behaviour occurs near the boundary because all the electrons turn round at the same point. If the plasma particles had a continuous velocity distribution this effect would disappear.

A parabolic potential well causes any particle, whatever its energy, to oscillate with the same fixed frequency. Even in a plasma with a continuous velocity distribution one would therefore find that there is no Landau damping, just as there is none in a mono-energetic plasma. In a well of more general shape particles of different energy do not have this synchronism, so that phase mixing and therefore Landau damping should occur.

Of course, if the plasma in the more general well is monoenergetic, all the electrons will follow the same trajectory and have the same period. Then one might expect that many of the properties of the parabolic well should still apply, and so they do.

Consider then a symmetrical potential well. In suitable dimensionless terms let the restoring force be $F(x)$ at position x, and let the undisturbed velocity in the two streams there be $\pm u$. With our previous notation we find the equations of motion for the two streams to be

$$\left(\frac{\partial}{\partial \tau} \pm u\frac{\partial}{\partial x}\right)^2 \xi_\pm - \xi_\pm \frac{dF}{dx} + \frac{\Lambda}{2u}(\xi_+ + \xi_-) = 0, \tag{109}$$

while the undisturbed velocity is given by

$$u(du/dx) = F. \tag{110}$$

If now we set
$$dx = ud\phi, \tag{111}$$

then ϕ is single valued as long as u does not change sign anywhere within the trap. Equations (109) and (110) combine to give

$$\left(\frac{\partial}{\partial \tau} \pm \frac{\partial}{\partial \phi}\right)^2 \xi_\pm - \frac{u''}{u}\xi_\pm + \frac{\Lambda}{2u}(\xi_+ + \xi_-) = 0. \tag{112}$$

The dimensionless co-ordinates can clearly be chosen so that $u = 0$, i.e. so that the plasma ends, at $\phi = \pm\frac{1}{2}\pi$.

The equation for a stationary even mode becomes

$$\Xi'' - \frac{u''}{u}\Xi + \frac{\Lambda}{u}\Xi = 0, \tag{113}$$

and once again we have to solve a standard eigenvalue problem. $\Lambda = 0$ is the lowest eigenvalue, and the corresponding even mode is evidently $\Xi = u$. Equation (113) also possess a discrete set of eigenvalues, of which those belonging to the even eigenfunctions describe the stationary even modes.

For an odd mode with small amplification rate σ, Θ is given by

$$\Theta'' - (u''/u)\Theta + \sigma^2\Theta = -2\sigma\Xi', \tag{114}$$

from which it follows that

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Theta u d\phi = -\frac{2}{\sigma}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi' u d\phi. \tag{115}$$

The equation
$$y'' - \mu_r(u''/u)\, y = 0 \tag{116}$$
has even solutions
$$y = c_{2r}(\phi)$$
for
$$\mu = \mu_{2r} \quad (r = 0, 1, 2, \ldots),$$

which vanish at $\phi = \pm\frac{1}{2}\pi$. Evidently $c_0(\phi) \equiv u$, and $\mu = 1$. On solving (114), with Θ expanded as a series in $c_{2r}(\phi)$ one finds that all terms have coefficients of order σ except for $c_0(\phi)$ whose coefficient is of order σ^{-1}. This is the only term which matters when σ is small. On carrying it back to the equation for Ξ one finds that a stationary odd mode is now described by

$$\Xi'' - \frac{u''}{u}\Xi + \frac{\Lambda}{u}\Xi = 4u' \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi' u \, d\phi \Big/ \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} u^2 \, d\phi. \tag{117}$$

The lowest eigensolution of this occurs for $\Lambda = \Lambda_1^* = 0$, and is $\Xi = \phi u$, as one easily can verify. Higher eigensolutions can be found by expanding Ξ in terms of the odd eigensolutions Ξ_{2r-1} of

$$\Xi'' - \frac{u''}{u}\Xi + \frac{\Lambda}{u}\Xi = 0. \tag{118}$$

The eigenvalues Λ^* of equation (117) are now found to be given by

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} u^2 \, d\phi + \sum_{r=0}^{\infty} \frac{4}{\Lambda^* - \Lambda_{2r-1}} \left\{ \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Xi_{2r-1} u' \, d\phi \right\}^2 = 0. \tag{119}$$

This equation is entirely analogous to the equation for Λ^* in the case of the parabolic well. The same type of shift in the odd eigenvalues will therefore occur.

Finally, the non-linear boundary régime remains essentially the same whether the mode is even or odd, for its nature depends only on the fact that the trapping field near the ends of the plasma varies linearly with distance. This is always approximately true, as long as the boundary layer remains thin enough.

All the properties of a monoenergetic plasma in a parabolic well should therefore have an analogy in symmetrical wells of more general shape.

The author first heard of this problem while visiting the Culham Laboratory as a consultant. He has had very interesting discussions about it with Professor W. B. Thompson, Dr G. Rowlands and Mr C. J. H. Watson.

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